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Gravitational corrections to the Euler-Heisenberg Lagrangian

Fiorenzo Bastianelli,[a,b] **José Manuel Dávila**[c] **and Christian Schubert**[a,c]

[a] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Mühlenberg 1, D-14476 Potsdam, Germany*

[b] *Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna, Italy*

[c] *Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Apdo. Postal 2-82, C.P. 58040, Morelia, Michoacán, México*

Abstract: We use the worldline formalism for calculating the one-loop effective action for the Einstein-Maxwell background induced by charged scalars or spinors, in the limit of low energy and weak gravitational field but treating the electromagnetic field nonperturbatively. The effective action is obtained in a form which generalizes the standard proper-time representation of the Euler-Heisenberg Lagrangian. We compare with previous work and discuss possible applications.

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Contents

1 Introduction

In 1936 Heisenberg and Euler derived their famous effective Lagrangian [1] describing the effect of a virtual electron-positron pair on an external Maxwell field in the one loop and constant field approximation. Its standard proper time representation is

$$\mathcal{L}_{\rm spin}(F) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, \mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]. \quad (1.1)$$

Here T is the proper-time of the loop fermion, m its mass, and a, b are the two invariants of the Maxwell field, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2-b^2 = B^2-E^2$, $\quad ab = \mathbf{E}\cdot\mathbf{B}$. The two subtraction terms implement the renormalization of charge and vacuum energy.

An analogous representation was found later for scalar QED [2, 3]:

$$\mathcal{L}_{\rm scal}(F) = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\, \mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\right]. \quad (1.2)$$

Although the effective Lagrangian for scalar QED is due to Weisskopf and Schwinger, for simplicity we will call it the "Scalar Euler-Heisenberg Lagrangian".

The Lagrangians (1.1), (1.2) historically provided the first examples for the concept of an effective Lagrangian, as well as the first nonperturbative results in quantum field theory. Despite of their formal simplicity they contain an enormous amount of physical information on low energy processes in QED. See [4–6] for reviews of their various applications and generalizations.

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The proper time integrals in eqs. (1.1), (1.2) can be done exactly in terms of certain special functions [6]. Alternatively, one can expand the integrands as power series in the field invariants, using the Taylor expansions

$$\frac{z}{\tanh(z)} = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}(2z)^{2n}, \tag{1.3}$$

$$\frac{z}{\sinh(z)} = -\sum_{n=0}^{\infty}\left(1-2^{1-2n}\right)\frac{B_{2n}}{(2n)!}(2z)^{2n}. \tag{1.4}$$

Here the B_{2n} are Bernoulli numbers. The terms in this expansion involving $N = 2n$ powers of the field contain the information on the low energy limit of the N photon scattering amplitudes, defined by all photon energies being small compared to the loop mass, $\omega_i \ll m, i = 1, \ldots, N$. In this limit the effective Lagrangian allows one to obtain these amplitudes in closed form and with moderate effort [7]. This should be contrasted with the fact that, away from the low energy limit, the calculation of these amplitudes for general N still presents a challenge. The four photon scattering amplitudes were obtained a long time ago [8], but the explicit calculation for the six-point case became possible only recently [9]. Beyond six points, results are essentially restricted to the massless case, where the amplitudes with N or $N-1$ equal helicities are known to vanish [10] and an explicit result has been obtained for $N-2$ equal helicities [11]. These massless N photon amplitudes are presently under intensive investigation (see the recent [12] and refs. therein). Regarding the off-shell case, to our knowledge even the four point amplitude is known only with maximally two legs off-shell [13].

Apart from the purely magnetic field case, the Euler-Heisenberg Lagrangians have also imaginary parts, induced by the poles which the integrands in (1.1), (1.2) have for $b \neq 0$. A simple application of Cauchy's theorem yields Schwinger's representation [3]

$$\begin{aligned}
\mathrm{Im}\mathcal{L}_{\mathrm{spin}}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\frac{\pi k}{\beta}\right], \\
\mathrm{Im}\mathcal{L}_{\mathrm{scal}}(E) &= \frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned} \tag{1.5}$$

with $\beta = \frac{eE}{m^2}$. These imaginary parts directly relate to the rates of electron-positron pair production by the electric field [3, 6]. The representation (1.5) makes it clear that this effect is nonperturbative in the field; its calculation requires the knowledge of the effective action to all orders in the weak field expansion.

Concerning higher loop corrections to the Lagrangians (1.1), (1.2) see [4, 14–16] for the spinor and [15–18] for the scalar case. For extensions to super QED see [19, 20].

In the present article, we will study the corrections to the Euler-Heisenberg Lagrangians (1.1), (1.2) due to an additional weak gravitational background. This amounts to calculating the one-loop effective actions for a generic Einstein-Maxwell background due to a scalar or spinor loop, to all orders in the electromagnetic field strength, and to leading order in the curvature.

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A sizable body of work exists already on the one-loop effective action in mixed gravitational-electromagnetic fields. Drummond and Hathrell in their seminal work [21] obtained the terms in the fermionic effective Lagrangian involving one curvature tensor and two field strength tensors:

$$\mathcal{L}_{\rm spin}^{(DH)} = \frac{1}{180(4\pi)^2 m^2}\Bigl(5RF_{\mu\nu}^2 - 26R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + 2R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2\Bigr) \tag{1.6}$$

(here and in the following we will often absorb the electric charge e into the field strength tensor F). The motivation of [21] for considering these terms was that they contain information on the modification of the photon dispersion relation by a generic gravitational background. While it is well-known that such modifications exist already in the pure QED case [5], the gravitational case is particularly interesting in that it permits superluminal propagation [22, 23], leading even to speculations on a possible violation of microcausality [24]. However, as emphasized in [25] these issues cannot be resolved at the level of the low-energy effective action since this would require information on the photon propagation in the full energy range.

As usual, a systematic computation of this effective action for either the scalar or spinor loop cases requires one to decide on the grouping of terms, the three basic options being

1. Summing over all derivatives on fields with the number of fields fixed.

2. Grouping together terms with a fixed mass dimension.

3. Fixing the number of derivatives and summing over the number of fields.

The first approach is usually called "derivative expansion". For our mixed electromagnetic-gravitational case, higher derivative corrections to the effective action (1.6) due to a scalar loop were considered in [26]. Those corrections can be summed up into "Barvinsky-Vilkovisky form factors", which are closed-form integral expressions involving Schwinger-parameter type integrals. See the recent [27] for the state-of-the-art of this approach.

The second one corresponds to the standard heat-kernel or "inverse mass" expansion. It is the most canonical one of the three in the sense that it is manifestly gauge and generally covariant order by order. The heat-kernel expansion of the one-loop effective action is usually written as

$$\Gamma[g,A] = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T} \int \frac{d^Dx\sqrt{g}}{(4\pi T)^{\frac{D}{2}}} \sum_{n=0}^{\infty} a_n(x) T^n \tag{1.7}$$

where D is the space-time dimension and $a_n(x)$ are the "heat-kernel coefficients". In $D = 4$ dimensions, the terms with $n = 0, 1, 2$ are UV divergent at $T = 0$, so that the corresponding coefficients are subject to renormalization. For our case of the Einstein-Maxwell background with a spin 0 or spin 1/2 loop the coefficients can, up to a_3, be obtained from more general results on the heat-kernel expansion [28, 29]. They are, for the scalar case[1]

$$\begin{aligned}
a_0 &= 1\,,\\
a_1 &= \Big(\frac{1}{6}-\xi\Big)R\,,\\
a_2 &= -\frac{1}{12}F_{\mu\nu}^2\,,\\
a_3 &= \frac{1}{360}\Big[5(6\xi-1)RF_{\mu\nu}^2+4R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}-6R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\\
&\qquad -2(\nabla^\alpha F_{\alpha\mu})^2-8(\nabla_\alpha F_{\mu\nu})^2-12F_{\mu\nu}\Box F^{\mu\nu}\Big]
\end{aligned}\tag{1.8}$$

and for the spinor case,

$$\begin{aligned}
a_0 &= -2\,,\\
a_1 &= \frac{1}{6}R\,,\\
a_2 &= -\frac{1}{3}F_{\mu\nu}^2\,,\\
a_3 &= \frac{1}{180}\Big[5RF_{\mu\nu}^2-4R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}-9R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\\
&\qquad +2(\nabla^\alpha F_{\alpha\mu})^2-7(\nabla_\alpha F_{\mu\nu})^2-18F_{\mu\nu}\Box F^{\mu\nu}\Big]\,.
\end{aligned}\tag{1.9}$$

Here the terms a_0, a_1, a_2 contribute to the renormalization of the vacuum energy, Newton's constant, and electric charge, respectively.

The expression (1.9) for the spinor case is equivalent to the one in (1.6), as can be seen by adding suitable total derivative terms; we will discuss this issue in section 5 below. The scalar case result (1.8) in this explicit form is new, as far as we know.

The third choice amounts to a generalization of the Euler-Heisenberg Lagrangian, the object of interest in this paper. Contrary to the pure QED case, for Einstein-Maxwell theory it is not obvious how one should define the effective Lagrangian for constant external fields, since the notion of constancy becomes ambiguous in curved space. To the best of our knowledge, the only previous attempt to treat the electromagnetic field and/or gravitational field nonperturbatively is due to Avramidi [30, 31]. This author generalizes the constancy of F to the covariant constancy of F and R,

$$\nabla_\alpha F_{\mu\nu} = \nabla_\alpha R_{\mu\nu\kappa\lambda} = 0\,.\tag{1.10}$$

For a background obeying (1.10) he obtains an Euler-Heisenberg type formula for the effective Lagrangian. However, the conditions (1.10) are rather strong, and imply, for example, also a consistency condition between F and R, since

$$\nabla_\alpha F_{\mu\nu}=0 \quad\rightarrow\quad [\nabla_\alpha,\nabla_\beta]F_{\mu\nu}=0 \quad\rightarrow\quad R_{\alpha\beta\mu\lambda}F^{\lambda}{}_{\nu}-R_{\alpha\beta\nu\lambda}F^{\lambda}{}_{\mu}=0\,.\tag{1.11}$$

[1] To obtain the scalar loop coefficients from appendix B of [29], replace $E\rightarrow -\xi R$ and $F_{ab}\rightarrow iF_{ab}$. Here the parameter ξ describes a non-minimal coupling to gravity. To obtain the spinor loop ones, replace $E\rightarrow -\frac{1}{4}R+\frac{i}{2}F_{ab}\gamma^a\gamma^b$ and $F_{ab}\rightarrow \frac{1}{4}R_{abcd}\gamma^c\gamma^d+iF_{ab}$.

This strongly suggests that the effective action for this special case can carry only some partial information on the low energy limit of the corresponding amplitudes, i.e. the one-loop one particle irreducible ("1PI") off-shell photon-graviton amplitudes involving a scalar or spinor loop. In the present paper, we adopt a more general definition of a curved-space Euler-Heisenberg Lagrangian ("EHL") by demanding that, like the QED EHL, it should contain the minimum set of terms in the covariant effective Lagrangian which would have the full information on the low-energy limit of the corresponding 1PI amplitudes. As usual in the graviton case, the amplitudes must be defined by linearizing gravity around flat Minkowski space. We will explicitly calculate the generalized EHL's for Einstein-Maxwell theory to linear order in the curvature, corresponding to the 1PI photon-graviton amplitudes with any number of photons but not more than one graviton. It is easily seen that this truncation corresponds to keeping all terms in the covariant effective Lagrangian which involve any number of electromagnetic field strength tensors, together with up to one factor of the curvature tensor, where this curvature tensor could also be replaced by two covariant derivatives. In this calculation, we use the recently completed extension of the worldline formalism [32–36] to curved space [37–41] made manifestly covariant by using Riemann normal coordinates and Fock-Schwinger gauge.

The structure of this paper is as follows. In section 2 we summarize the worldline algorithm for the calculation of one-loop effective actions in mixed gravitational-electromagnetic fields. The calculation of the generalized Euler-Heisenberg Lagrangian is presented in 3 for the scalar and in 4 for the spinor loop case. In 5 we compare with previous work and discuss possible applications of these Lagrangians. We summarize our findings in section 6. Our differential geometry conventions are given in appendix A, where we also collect some useful formulas. In appendix B we discuss some properties of the worldline Green's functions in a constant field, to be introduced below.

2 Worldline representation of the effective action in Einstein-Maxwell theory

Let us start with the (euclidean) effective action for a complex scalar field ϕ coupled to electromagnetism and gravity,

$$S[\phi,\phi^*;g,A] = -\int d^Dx\sqrt{g}\Big[g^{\mu\nu}(\partial_\mu - ieA_\mu)\phi^*(\partial_\nu + ieA_\nu)\phi + (m^2+\xi R)\phi^*\phi\Big] \tag{2.1}$$

where ξ describes an additional non-minimal coupling to the scalar curvature R. Quantization produces the following effective action ($Z[g,A] = e^{\Gamma[g,A]} = \int \mathcal{D}\phi\mathcal{D}\phi^*\ e^{S[\phi,\phi^*;g,A]}$)

$$\Gamma[g,A] = \ln\det{}^{-1}(-\square_A + m^2 + \xi R) = -\mathrm{Tr}\ln(-\square_A + m^2 + \xi R) \tag{2.2}$$

where $\square_A$ is the gauge and gravitational covariant laplacian for scalar fields. It can be represented by the following worldline path integral (see, e.g., [37, 38])[2]

$$\Gamma[g,A] = \int_0^\infty \frac{dT}{T}\int_{PBC}\mathcal{D}x\ e^{-S[x^\mu;g,A]} \tag{2.3}$$

[2]Note that our definition of the global sign of the effective action follows [36] rather than [38, 42]. It corresponds to a euclidean tree level action $\Gamma = -\int d^4x\sqrt{g}\,\frac{1}{4}F_{\mu\nu}^2$.

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where

$$S[x^\mu; g, A] = \int_0^1 d\tau \left(\frac{1}{4T} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + ieA_\mu(x)\dot{x}^\mu + T(\xi R(x) + m^2) \right) . \tag{2.4}$$

Here T is the proper-time of the loop particle, and the path integral is to be performed over all closed loops in spacetime $x(\tau)$ with periodic boundary conditions $x(1) = x(0)$.

Following the "string-inspired" procedure [32–35] we will evaluate the path integral $\int \mathcal{D}x(\tau)$ by manipulating it into gaussian form, using a double expansion. First, one Taylor expands the external fields at some point x_0 [35, 43–46]. This is most conveniently done in covariant form, i.e., using a combination of Fock-Schwinger gauge and Riemann normal coordinates [46, 47]:

$$g_{\mu\nu}(x = x_0 + y) = g_{\mu\nu}(x_0) + \frac{1}{3} R_{\mu\alpha\beta\nu}(x_0)\, y^\alpha\, y^\beta + \cdots \tag{2.5}$$

$$\begin{aligned} A_\mu(x = x_0 + y) = &-\frac{1}{2} F_{\mu\nu}(x_0)\, y^\nu - \frac{1}{3} F_{\mu\nu;\alpha}(x_0)\, y^\nu\, y^\alpha - \frac{1}{8}\Big[F_{\mu\nu;\alpha\beta}(x_0) \\ &+ \frac{1}{3} R_{\alpha\mu}{}^{\lambda}{}_{\beta}(x_0) F_{\lambda\nu}(x_0) \Big] y^\alpha\, y^\beta\, y^\nu + \cdots \end{aligned} \tag{2.6}$$

(see appendix A for our Riemannian geometry conventions). The worldline action then takes the form

$$S[x^\mu; R, F] = \frac{1}{4T} \int_0^1 d\tau\, \dot{y}^\mu(\tau) g_{\mu\nu}(x_0) \dot{y}^\nu(\tau) + S_{\text{int}}[x^\mu; R, F] \tag{2.7}$$

where $S_{\text{int}}[x^\mu; R, F]$ contains an infinite number of interaction terms. In principle, all these interaction exponentials are to be expanded out, although realistically the arising multiple series has to be truncated to some desired level. Usually this truncation will be either in the number of fields, in the number of derivatives, or in the mass dimensions.

Next, one has to fix the zero mode of the path integral, due to translation invariance in spacetime. There are two standard ways of doing this, both using a restriction of the path integration to fluctuations around the expansion point x_0,

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau), \tag{2.8}$$

where the path integral measure is split into

$$Dx = \frac{d^D x_0 \sqrt{g(x_0)}}{(4\pi T)^{\frac{D}{2}}}\, Dy\ . \tag{2.9}$$

The choice is in the constraints imposed on $y(\tau)$, which are either Dirichlet boundary conditions

$$y(0) = y(1) = 0 \tag{2.10}$$

meaning that x_0 is on the loop ("DBC scheme"), or the "string-inspired" condition ("SI scheme")

$$\int_0^1 d\tau\, y^\mu(\tau) = 0 \tag{2.11}$$

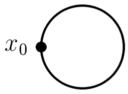

Figure 1. DBC scheme.

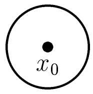

Figure 2. SI scheme.

which makes x_0 the center of mass of the loop, see figures 1 and 2.

The DBC scheme leads to a worldline propagator

$$\langle y^\mu(\tau) y^\nu(\sigma)\rangle = -2T g^{\mu\nu}(x_0)\Delta(\tau,\sigma) \tag{2.12}$$

where

$$\begin{aligned}\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty}\left[-\frac{2}{\pi^2 m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] \\ &= (\tau-1)\sigma\,\theta(\tau-\sigma) + (\sigma-1)\tau\,\theta(\sigma-\tau)\,.\end{aligned} \tag{2.13}$$

The propagator in the SI scheme is

$$\langle y^\mu(\tau) y^\nu(\sigma)\rangle = -T g^{\mu\nu}(x_0) G_B(\tau,\sigma) \tag{2.14}$$

where

$$G_B(\tau,\sigma) = 2\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}\frac{\mathrm{e}^{2i\pi m(\tau-\sigma)}}{(2i\pi m)^2} = |\tau-\sigma| - \Big(\tau-\sigma\Big)^2 - \frac{1}{6}\,. \tag{2.15}$$

For gauge theory in flat space, either propagator can be chosen for a straightforward perturbative calculation of the one-loop effective action via formal gaussian integration.[3] The effective Lagrangian obtained in the DBC scheme coincides with the heat kernel result, while the SI scheme differs from it, but only by total derivative terms [36, 43, 44]. Thus both schemes are completely equivalent, but the SI scheme is computationally preferable, since it preserves the translation invariance in the proper-time.

Proceeding to the inclusion of gravitational backgrounds, here a number of mathematical difficulties arise which are not present in flat space, starting with the observation that the structure of the worldline Lagrangian (2.4) generically leads to ill-defined expressions involving, e.g., $\delta(0), (\delta(\tau-\sigma))^2, \ldots$ in a gaussian integration. A completely satisfactory formalism for dealing with these issues has emerged only in recent years [48–53]. Here we can only sketch the procedure; a brief discussion appears in [54] and all the details can be found in [37].

First, in curved space the path integral measure is nontrivial. Following [48, 49] we exponentiate it as follows,

$$\mathcal{D}x = Dx \prod_{0\le\tau<1}\sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{PBC} DaDbDc\,\mathrm{e}^{-S_{gh}[x,a,b,c]}, \tag{2.16}$$

[3] In flat space calculations the constant $-\frac{1}{6}$ in G_B does not affect physical quantities and is therefore usually omitted.

with a ghost action

$$S_{gh}[x,a,b,c] = \int_0^1 d\tau\, \frac{1}{4T} g_{\mu\nu}(x) \Big[a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big]\,. \tag{2.17}$$

After the replacement of $g_{\mu\nu}(x)$ by its normal coordinate expansion (2.5) the correlators of these ghost fields just involve δ functions,

$$\begin{aligned}\langle a^\mu(\tau)a^\nu(\sigma)\rangle &= 2T g^{\mu\nu}(x_0)\delta(\tau-\sigma),\\ \langle b^\mu(\tau)c^\nu(\sigma)\rangle &= -4T g^{\mu\nu}(x_0)\delta(\tau-\sigma)\,.\end{aligned} \tag{2.18}$$

The ghost field contributions will cancel all ill-defined divergent terms of the type mentioned above, arising from the Wick contractions of the coordinate fields. This cancellation of infinities generally still leaves integrals with ambiguities. A basic example is

$$\int_0^1 d\tau \int_0^1 d\sigma\, \delta(\tau-\sigma)\theta(\tau-\sigma)\theta(\sigma-\tau)\,. \tag{2.19}$$

This type of integral requires a regularization, and different regularizations will assign different finite values to it [37].

From the point of view of one-dimensional quantum field theory, we are dealing here with a theory which, without the terms from the nontrivial path integral measure, would be UV divergent but super-renormalizable. Including those terms removes all divergences, but leaves finite ambiguities, so that agreement with standard spacetime QFT is reached only after adding a finite number of counterterms with finite, regularization-dependent coefficients. The method which we will adopt here is (one-dimensional) dimensional regularization [52, 53], since it is presently the only known regulator which preserves the general covariance. It needs only a single counterterm proportional to the curvature scalar, $-R/4$ in the present notations. Therefore in this scheme the only effect of the spurious UV divergences is a change of the parameter ξ in the worldline Lagrangian (2.4) into $\bar\xi$,

$$\bar\xi := \xi - \frac{1}{4}\,. \tag{2.20}$$

A further subtlety shows up if one wishes to combine the SI scheme with the use of the Riemann-Fock-Schwinger expansion (2.5), (2.6). As is well-known, this expansion is useful only for the calculation of covariant quantities; when applied to non-covariant quantities, it yields a result which is formally covariant but correct only in Riemann normal coordinates. The DBC scheme in curved space still yields the same effective Lagrangian as the standard heat-kernel method, and thus also guarantees covariance. This does not extend to the SI scheme, since it turns out that the total derivative terms by which the SI effective Lagrangian differs from the DBC one generally are not covariant [38, 55]. A solution to this problem convenient for actual calculations was found in [39]. There it was shown that, using Riemann normal coordinates from the beginning and performing a BRST treatment of the symmetry corresponding to a shift of x_0, the difference between the two effective Lagrangians can be reduced to manifestly covariant terms. This is achieved by the addition of further Fadeev-Popov type terms to the worldline Lagrangian in the "string-inspired" scheme. Those terms are infinite in number but easy to determine order by order. In our present approximation, this Fadeev-Popov action can be truncated as

$$S_{FP} = -\bar{\eta}_\mu \int_0^1 d\tau\, Q^\mu_\nu(x_0, y(\tau))\, \eta^\nu \tag{2.21}$$

where

$$Q^\mu{}_\nu(x_0, y) = \delta^\mu_\nu + \frac{1}{3} R^\mu{}_{\alpha\beta\nu}(x_0)\, y^\alpha y^\beta + \ldots . \tag{2.22}$$

The propagator for the (constant) ghost fields $\eta, \bar{\eta}$ is trivial,

$$\langle \eta^\mu \bar{\eta}_\nu \rangle = -\delta^\mu_\nu \ . \tag{2.23}$$

Having concluded our discussion of the scalar loop case, we proceed to the case of a spin 1/2 particle in the loop. The euclidean action for a Dirac field Ψ coupled to electromagnetism (A_μ) and gravity ($e_\mu{}^a$) is given by

$$S[\Psi, \bar{\Psi}; e, A] = -\int d^D x\; e\, \bar{\Psi}(\not{\nabla} + m)\Psi \tag{2.24}$$

where $e_\mu{}^a$ is the vielbein, $e = \det e_\mu{}^a$, $\omega_{\mu ab}$ is the spin connection, and

$$\not{\nabla} = \gamma^a e_a{}^\mu \nabla_\mu \,, \qquad \nabla_\mu = \partial_\mu + ieA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \ . \tag{2.25}$$

The effective action depends on the background fields $e_\mu{}^a$ and A_μ, and formally reads as ($e^{\Gamma[e,A]} \equiv \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\; e^{S[\Psi,\bar{\Psi};e,A]} = \mathrm{Det}(\not{\nabla} + m)$)

$$\begin{aligned}
\Gamma[e, A] &= \ln \mathrm{Det}(\not{\nabla} + m) = \ln[\mathrm{Det}(\not{\nabla} + m)\mathrm{Det}(-\not{\nabla} + m)]^{\frac{1}{2}} \\
&= \frac{1}{2}\mathrm{Tr}\ln(-\not{\nabla}^2 + m^2) \\
&= \frac{1}{2}\mathrm{Tr}\ln\Big(-\Box_A + m^2 + \frac{1}{4}R\Big) \ .
\end{aligned} \tag{2.26}$$

A worldline path integral representation for this effective action can be written in a manifestly local Lorentz invariant way [40] (i.e. in terms of the metric rather than the vielbein)

$$\Gamma[g, A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} \int_{PBC} \mathcal{D}x \int_{ABC} \mathcal{D}\psi\; e^{-S[x^\mu, \psi^\mu; g, A]} \tag{2.27}$$

with

$$\begin{aligned}
S[x^\mu, \psi^\mu; g, A] = \int_0^1 d\tau \bigg[&\frac{1}{4T} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + ieA_\mu(x)\dot{x}^\mu + T\Big(\frac{1}{4}R(x) + m^2\Big) \\
&+ \frac{1}{2T}\Big(g_{\mu\nu}(x)\psi^\mu\dot{\psi}^\nu - \partial_\mu g_{\nu\lambda}(x)\psi^\mu\psi^\nu\dot{x}^\lambda\Big) - ieF_{\mu\nu}(x)\psi^\mu\psi^\nu\bigg] \ .
\end{aligned} \tag{2.28}$$

Note that the bosonic term appearing in the first line is the same as for a scalar particle with $\xi = \frac{1}{4}$, see eq. (2.4) and (2.20), while the second term contains the worldline fermions and describes the dependence on the spin of the particle. This action also makes it clear that there are only linear couplings of the spin 1/2 particle to the metric $g_{\mu\nu}$. The worldline fields $\psi^\mu(\tau)$ are Grassmann valued and antiperiodic, $\psi(1) = -\psi(0)$. The free spin path integral is normalized as $2^{D/2}$. (Note also that our ψ^μ corresponds to $\sqrt{T}\psi^\mu$ in the conventions of [36].)

Again we gaussianize the double path integral in (2.27) by the use of the Riemann-Fock-Schwinger expansion (2.5), (2.6). Now also the expansion of $F_{\mu\nu}$ is needed, which follows from (2.6):

$$\begin{aligned} F_{\mu\nu}(x_0+y) &= F_{\mu\nu}(x_0) + F_{\mu\nu;\alpha}(x_0)\, y^\alpha + \frac{1}{2}F_{\mu\nu;\alpha\beta}(x_0)\, y^\alpha\, y^\beta \\ &\quad + \frac{1}{6}\Big(R_{\alpha\mu}{}^{\lambda}{}_{\beta}(x_0)F_{\lambda\nu}(x_0) + R_{\alpha\nu}{}^{\lambda}{}_{\beta}(x_0)F_{\mu\lambda}(x_0)\Big)\, y^\alpha\, y^\beta + \dots \end{aligned} \tag{2.29}$$

Eq. (2.7) generalizes to

$$\begin{aligned} S[x^\mu,\psi^\nu;R,F] &= \frac{1}{T}\int_0^1 d\tau\, \left[\frac{1}{4}\dot{y}^\mu(\tau)g_{\mu\nu}(x_0)\dot{y}^\nu(\tau) + \frac{1}{2}\psi^\mu(\tau)g_{\mu\nu}(x_0)\dot{\psi}^\nu(\tau)\right] \\ &\quad + S_{\mathrm{int}}[x^\mu,\psi^\nu;R,F]\,. \end{aligned} \tag{2.30}$$

The propagator of the worldline fermions then becomes

$$\langle\psi^\mu(\tau)\psi^\nu(\sigma)\rangle = \frac{1}{2}T g^{\mu\nu}(x_0)G_F(\tau,\sigma) \tag{2.31}$$

where

$$G_F(\tau,\sigma) = 2\sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{i\pi(2m+1)(\tau-\sigma)}}{i\pi(2m+1)} = \mathrm{sign}(\tau-\sigma)\,. \tag{2.32}$$

Note that, due to the antisymmetry of the spin path integral, there is no zero mode and thus no related ambiguity for this propagator.

Like the bosonic path integral measure, the fermionic one is nontrivial in curved space, leading to a generalization of (2.16) to

$$\mathcal{D}x\mathcal{D}\psi = Dx D\psi \int_{PBC} DaDbDc \int_{ABC} D\alpha\; \mathrm{e}^{-S_{gh}[x,a,b,c,\alpha]}, \tag{2.33}$$

where the ghost action now is

$$S_{gh}[x,a,b,c,\alpha] = \int_0^1 d\tau\; \frac{1}{4T} g_{\mu\nu}(x)\Big[a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau) + 2\alpha^\mu(\tau)\alpha^\nu(\tau)\Big]\,. \tag{2.34}$$

The correlator of the new ghost field $\alpha^\mu(\tau)$ is, after the normal coordinate expansion,

$$\langle\alpha^\mu(\tau)\alpha^\nu(\sigma)\rangle = T g^{\mu\nu}(x_0)\delta(\tau-\sigma)\,. \tag{2.35}$$

Again there are cancellations of ill-defined divergent terms between the ψ and the α path integrals, forcing one to choose a regularization and possibly leading to a modification of the counterterms introduced for the spinless worldline Lagrangian above. However, it turns out that in dimensional regularization this does not happen; the sole counterterm

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$-\frac{1}{4}R$ remains also the correct one for the spin 1/2 case [40]. Its effect is just to remove the term linear in R which was there in the initial worldline Lagrangian (2.28). Other regularizations have been discussed in [56].

In principle, this is all one has to know for calculating the one-loop effective action for spin 0 or spin 1/2 particle in the Einstein-Maxwell background, or the corresponding amplitudes [38, 40]. However, since we are aiming at a result which is nonperturbative in the electromagnetic field, for us the following modification will be essential: Note that the leading terms of the Fock-Schwinger expansions (2.6) and (2.29) yield terms in $S_{\rm int}$ which are quadratic in the worldline fields. Thus instead of using them in the interaction part one can absorb them in the worldline propagators. In the SI scheme this leads to the following change of the correlators (2.14), (2.31) [15, 57, 58],

$$\begin{aligned} \langle y^\mu(\tau)y^\nu(\sigma)\rangle &= -T\mathcal{G}_B^{\mu\nu}(\tau,\sigma)\,, \\ \langle \psi^\mu(\tau)\psi^\nu(\sigma)\rangle &= \frac{1}{2}T\mathcal{G}_F^{\mu\nu}(\tau,\sigma)\,. \end{aligned} \tag{2.36}$$

The new worldline propagators are trigonometric functions of the field strength matrix F, and thus, in general, nontrivial Lorentz matrices:

$$\begin{aligned} \mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2) &= \left[\frac{1}{2(FT)^2}\left(\frac{FT}{\sin(FT)}\mathrm{e}^{-iFT\dot{G}_{B12}}+iFT\dot{G}_{B12}-1\right)\right]^{\mu\nu}, \\ \mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2) &= \left[G_{F12}\frac{\mathrm{e}^{-iFT\dot{G}_{B12}}}{\cos(FT)}\right]^{\mu\nu}. \end{aligned} \tag{2.37}$$

Here and in the following we abbreviate $G_{B12} = G_B(\tau_1,\tau_2)$ etc., and a 'dot' on a Green's function denotes a derivative with respect to the first variable. In the computation of the power series appearing in the definitions (2.37) it should be understood that indices are raised and lowered with the metric $g_{\mu\nu}(x_0)$.

Finally, the free gaussian path integrals get also modified and become field-dependent; namely, the coordinate path integral acquires a factor of $\det^{-\frac{1}{2}}[\sin(FT)/FT]$, the spin path integral a $\det^{\frac{1}{2}}[\cos(FT)]$. By themselves these factors just reproduce the integrands of the (unrenormalized) Euler-Heisenberg Lagrangians (1.1), (1.2).

This version of the formalism has already been applied extensively to the calculation of pure QED amplitudes or effective actions in a constant background field [15, 16, 45, 59, 60]. More recently it has been used for a first calculation of the photon-graviton polarization tensor in a constant field [42]. See also [15, 61–63] for an extension to the nonabelian case.

3 Calculation of the effective Lagrangian: scalar loop

In the following we specialize to the SI scheme. According to the above, the worldline representation of the effective Lagrangian for the scalar loop in $D = 4$ dimensions in this scheme can be written as[4]

$$\mathcal{L}_{\rm scal} = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}{\det}^{-\frac{1}{2}}\left[\frac{\sin(FT)}{FT}\right]\left\langle \mathrm{e}^{-S_{\rm int}[x^\mu,a,b,c,\eta;R,F]}\right\rangle. \tag{3.1}$$

[4] In the following it is understood that all spacetime fields are sitting at the expansion point x_0, and $\Gamma = \int d^4x_0\sqrt{g}\mathcal{L}$.

In our one-graviton approximation, the worldline interaction Lagrangian can be truncated as

$$
\begin{aligned}
S_{\rm int} &= S_{grav} + S_{em} + S_{em,grav} + S_{gh} + S_{FP}\,, && (3.2)\\
S_{grav} + S_{gh} &= T\bar{\xi}\,R + \frac{1}{12T}\int_0^1 d\tau R_{\mu\alpha\beta\nu} y^\alpha y^\beta \Big[\dot{y}^\mu \dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu\Big], && (3.3)\\
S_{em} &= \int_0^1 d\tau \Big[-\frac{i}{3} F_{\mu\nu;\alpha} \dot{y}^\mu y^\nu y^\alpha - \frac{i}{8} F_{\mu\nu;\alpha\beta}\, \dot{y}^\mu\, y^\nu\, y^\alpha\, y^\beta\Big], && (3.4)\\
S_{em,grav} &= -\frac{i}{24}\int_0^1 d\tau R_{\alpha\mu}{}^\lambda{}_\beta F_{\lambda\nu} y^\nu\, y^\alpha\, y^\beta\, \dot{y}^\mu, && (3.5)\\
S_{FP} &= -\frac{1}{3}\int_0^1 d\tau\ \bar{\eta}_\mu R^\mu{}_{\alpha\beta\nu}\, y^\alpha y^\beta\, \eta^\nu. && (3.6)
\end{aligned}
$$

Note that the term involving $y^\mu F_{\mu\nu}\dot{y}^\nu$ has been omitted from $S_{\rm em}$.

For easy reference, let us also list the complete set of worldline propagators of the SI scheme:

$$
\begin{aligned}
\langle y^\mu(\tau) y^\nu(\sigma)\rangle &= -T\mathcal{G}_B^{\mu\nu}(\tau,\sigma)\,,\\
\langle a^\mu(\tau) a^\nu(\sigma)\rangle &= 2T g^{\mu\nu}\delta(\tau-\sigma)\,,\\
\langle b^\mu(\tau) c^\nu(\sigma)\rangle &= -4T g^{\mu\nu}\delta(\tau-\sigma)\,,\\
\langle \eta^\mu \bar{\eta}_\nu\rangle &= -\delta^\mu_\nu\,, && (3.7)
\end{aligned}
$$

where $\mathcal{G}_B$ was given in (2.37).

With all this machinery in place, it is then straightforward to obtain the following result for the (unrenormalized) scalar loop effective Lagrangian in the one-graviton approximation,

$$
\begin{aligned}
\mathcal{L}^{(SI)}_{\rm scal} = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\, {\rm e}^{-m^2T} {\rm det}^{-1/2}\left[\frac{\sin(FT)}{FT}\right] \Bigg\{ & 1 - T\bar{\xi}R + \frac{T}{3}\mathcal{G}^{\alpha\beta}_{B11} R_{\alpha\beta}\\
& + \frac{iT^2}{8} F_{\mu\nu;\alpha\beta}\, \dot{\mathcal{G}}^{\mu\nu}_{B11}\, \mathcal{G}^{\alpha\beta}_{B11} + \frac{i}{8}T^2\left(F_{\mu\nu;\beta\alpha} + F_{\mu\nu;\alpha\beta}\right) \dot{\mathcal{G}}^{\mu\beta}_{B11} \mathcal{G}^{\nu\alpha}_{B11}\\
& - \frac{iT^2}{24} F_{\lambda\nu} R^\lambda{}_{\alpha\beta\mu} \left(\dot{\mathcal{G}}^{\nu\mu}_{B11}\, \mathcal{G}^{\alpha\beta}_{B11} + \dot{\mathcal{G}}^{\alpha\mu}_{B11}\, \mathcal{G}^{\nu\beta}_{B11} + \dot{\mathcal{G}}^{\beta\mu}_{B11}\, \mathcal{G}^{\nu\alpha}_{B11}\right)\\
& + \frac{T}{12} R_{\mu\alpha\beta\nu}\left(\dot{\mathcal{G}}^{\mu\alpha}_{B11} \dot{\mathcal{G}}^{\beta\nu}_{B11} + \dot{\mathcal{G}}^{\mu\beta}_{B11} \dot{\mathcal{G}}^{\alpha\nu}_{B11} + \left(\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2g^{\mu\nu}\delta(0)\right)\mathcal{G}^{\alpha\beta}_{B11}\right)\\
& - \frac{T^3}{6} F_{\alpha\beta;\gamma} F_{\mu\nu;\delta}\int_0^1 d\tau_1 \left(\dot{\mathcal{G}}^{\alpha\nu}_{B12}\, \dot{\mathcal{G}}^{\beta\mu}_{B12}\, \mathcal{G}^{\gamma\delta}_{B12} + \dot{\mathcal{G}}^{\alpha\nu}_{12}\, \mathcal{G}^{\beta\delta}_{12}\, \dot{\mathcal{G}}^{\gamma\mu}_{B12}\right)\Bigg\}. \qquad (3.8)
\end{aligned}
$$

Here in the last term it is understood that $\tau_2 = 0$. Although getting (3.8) from (3.1) is a matter of standard combinatorics, a few comments are in order:

1. In the next-to-last term in braces the $\delta(0)$ comes from the ghost sector and substracts a $\delta(0)$ contained in $\ddot{\mathcal{G}}_{B11}$.

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2. In the last term in braces the Wick contractions produce also terms involving a contracting among the fields inside one factor of $F_{\mu\nu;\alpha}\dot{y}^\mu y^\nu y^\alpha$, however those have vanishing τ_1 or τ_2 integrals due to (B.5). The remaining terms have been reduced to a minimal set using integrations by parts in τ_1 and the Bianchi identity (A.6).

3. The third term in braces comes from the Fadeev-Popov part S_{FP} of the worldline action. The inclusion of this term is necessary to obtain the equivalence with the standard heat kernel result, to be shown in section 5 below. This confirms the formal reasoning of [39].

4. No ambiguous integrals are encountered yet at the level of our present calculation, so that regularization was not really necessary. This can be understood from the fact that any arising ambiguity would have to be cancelled by regularization dependent counterterms. Those generally involve products of Christoffel symbols [37], and can therefore in Riemann normal coordinates appear only starting at the quadratic level in the curvature.

4 Calculation of the effective Lagrangian: spinor loop

For the spinor loop, the analogue of (3.1) is

$$\mathcal{L}_{\rm spin} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}{\rm det}^{-\frac{1}{2}}\Big[\frac{\tan(FT)}{FT}\Big]\Big\langle \mathrm{e}^{-S_{\rm int}[x^\mu,\psi^\mu,a,b,c,\alpha,\eta;R,F]}\Big\rangle . \tag{4.1}$$

The various components of the worldline interaction Lagrangian (3.2) generalize as follows:

$$\begin{aligned} S_{grav}+S_{gh} &= \frac{1}{T}\int_0^1 d\tau\bigg\{\frac{1}{12}R_{\mu\alpha\beta\nu}y^\alpha y^\beta\Big[\dot{y}^\mu\dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu + 2\alpha^\mu\alpha^\nu\Big] \\ &\quad +\frac{1}{6}R_{\mu\alpha\beta\nu}\,y^\alpha\,y^\beta\,\psi^\mu\,\dot{\psi}^\nu + \frac{1}{6}(R_{\mu\alpha\lambda\beta}+R_{\mu\beta\lambda\alpha})\dot{y}^\alpha\,y^\lambda\,\psi^\mu\,\psi^\beta\bigg\}, && (4.2)\\ S_{em} &= \int_0^1 d\tau\Big[-\frac{i}{3}F_{\mu\nu;\alpha}\Big(\dot{y}^\mu y^\nu + 3\psi^\mu\psi^\nu\Big)y^\alpha - \frac{i}{8}F_{\mu\nu;\alpha\beta}\Big(\dot{y}^\mu y^\nu + 4\psi^\mu\psi^\nu\Big)y^\alpha y^\beta\Big], && (4.3)\\ S_{em,grav} &= -\frac{i}{24}\int_0^1 d\tau R_{\alpha\mu}{}^\lambda{}_\beta F_{\lambda\nu}\Big[\dot{y}^\mu\,y^\nu + 8\psi^\mu\psi^\nu\Big]\,y^\alpha\,y^\beta\,. && (4.4)\end{aligned}$$

S_{FP} is not modified from its form for the spinless case, eq. (3.6). In addition to the propagators of the scalar case, (3.7), we have now also

$$\begin{aligned}\langle\psi^\mu(\tau)\psi^\nu(\sigma)\rangle &= \frac{1}{2}T\mathcal{G}_F^{\mu\nu}(\tau,\sigma)\,,\\ \langle\alpha^\mu(\tau)\alpha^\nu(\sigma)\rangle &= Tg^{\mu\nu}\delta(\tau-\sigma)\,. \end{aligned}\tag{4.5}$$

The final result for the spinor loop case becomes

$$
\begin{aligned}
\mathcal{L}^{(SI)}_{\text{spin}} &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\det{}^{-1/2}\left[\frac{\tan(FT)}{FT}\right] \\
&\quad\times\Biggl\{1+\frac{iT^2}{8}F_{\mu\nu;\alpha\beta}\,\mathcal{G}^{\alpha\beta}_{B11}\Bigl(\dot{\mathcal{G}}^{\mu\nu}_{B11}-2\,\mathcal{G}^{\mu\nu}_{F11}\Bigr) \\
&\quad+\frac{iT^2}{8}\left(F_{\mu\nu;\beta\alpha}+F_{\mu\nu;\alpha\beta}\right)\dot{\mathcal{G}}^{\mu\beta}_{B11}\mathcal{G}^{\nu\alpha}_{B11}+\frac{T}{3}R_{\alpha\beta}\,\mathcal{G}^{\alpha\beta}_{B11} \\
&\quad-\frac{iT^2}{24}F_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu}\,\Bigl(\dot{\mathcal{G}}^{\nu\mu}_{B11}\,\mathcal{G}^{\alpha\beta}_{B11}+\dot{\mathcal{G}}^{\alpha\mu}_{B11}\,\mathcal{G}^{\nu\beta}_{B11}+\dot{\mathcal{G}}^{\beta\mu}_{B11}\,\mathcal{G}^{\nu\alpha}_{B11}+4\,\mathcal{G}^{\mu\nu}_{F11}\,\mathcal{G}^{\alpha\beta}_{B11}\Bigr) \\
&\quad+\frac{T}{12}R_{\mu\alpha\beta\nu}\Bigl(\dot{\mathcal{G}}^{\mu\alpha}_{B11}\dot{\mathcal{G}}^{\beta\nu}_{B11}+\dot{\mathcal{G}}^{\mu\beta}_{B11}\dot{\mathcal{G}}^{\alpha\nu}_{B11}+\Bigl(\ddot{\mathcal{G}}^{\mu\nu}_{B11}-2g^{\mu\nu}\delta(0)\Bigr)\mathcal{G}^{\alpha\beta}_{B11} \\
&\quad+\dot{\mathcal{G}}^{\alpha\beta}_{B11}\,\mathcal{G}^{\mu\nu}_{F11}+\dot{\mathcal{G}}^{\nu\beta}_{B11}\,\mathcal{G}^{\mu\alpha}_{F11}-\mathcal{G}^{\alpha\beta}_{B11}\,\Bigl(\dot{\mathcal{G}}^{\mu\nu}_{F11}-2g^{\mu\nu}\delta(0)\Bigr)\Bigr) \\
&\quad-\frac{1}{6}T^3F_{\alpha\beta;\gamma}\,F_{\mu\nu;\delta}\,\int_0^1 d\tau_1\Bigl(\dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\dot{\mathcal{G}}^{\beta\mu}_{B12}\,\mathcal{G}^{\gamma\delta}_{B12}+\dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\mathcal{G}^{\beta\delta}_{B12}\,\dot{\mathcal{G}}^{\gamma\mu}_{B12} \\
&\quad+\frac{3}{2}\,\mathcal{G}^{\gamma\delta}_{B12}\,\mathcal{G}^{\alpha\mu}_{F12}\,\mathcal{G}^{\beta\nu}_{F12}\Bigr)\Biggr\}
\end{aligned}
\tag{4.6}
$$

where again $\tau_2 = 0$.

5 Comparison with previous results

As a check on our effective Lagrangians (3.8), (4.6), let us extract the terms corresponding to the heat kernel coefficients a_3. This can be easily done using formulas (B.12), and yields, after performing the global proper-time integration,

$$
\begin{aligned}
\mathcal{L}^{(SI)}_{\text{scal}} &= \frac{1}{16\pi^2}\frac{e^2}{m^2}\left[\frac{1}{12}(\bar{\xi}+\frac{1}{12})RF^2_{\mu\nu}+\frac{1}{180}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\right. \\
&\qquad\left.-\frac{1}{72}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}-\frac{1}{180}(\nabla_\alpha F_{\mu\nu})^2-\frac{1}{72}F_{\mu\nu}\Box F^{\mu\nu}\right], \\
\mathcal{L}^{(SI)}_{\text{spin}} &= -\frac{1}{8\pi^2}\frac{e^2}{m^2}\left[-\frac{1}{72}RF^2_{\mu\nu}+\frac{1}{180}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\right. \\
&\qquad\left.+\frac{1}{36}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}-\frac{1}{180}(\nabla_\alpha F_{\mu\nu})^2+\frac{1}{36}F_{\mu\nu}\Box F^{\mu\nu}\right].
\end{aligned}
\tag{5.1}
$$

Here the identities (A.3)–(A.5) have been used to combine some terms.

This is different from the heat kernel results (1.8), (1.9), which read, after the T

integration,

$$\begin{aligned}
\mathcal{L}^{(HK)}_{\text{scal}} &= \frac{1}{16\pi^2}\frac{e^2}{m^2}\Bigg[\frac{1}{12}\left(\bar{\xi}+\frac{1}{12}\right)RF^2_{\mu\nu}+\frac{1}{90}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} \\
&\quad -\frac{1}{60}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}-\frac{1}{45}(\nabla_\alpha F_{\mu\nu})^2-\frac{1}{30}F_{\mu\nu}\Box F^{\mu\nu}-\frac{1}{180}(\nabla^\alpha F_{\alpha\mu})^2\Bigg], \\
\mathcal{L}^{(HK)}_{\text{spin}} &= -\frac{1}{8\pi^2}\frac{e^2}{m^2}\Bigg[-\frac{1}{72}RF^2_{\mu\nu}+\frac{1}{90}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} \\
&\quad +\frac{1}{40}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}+\frac{7}{360}(\nabla_\alpha F_{\mu\nu})^2+\frac{1}{20}F_{\mu\nu}\Box F^{\mu\nu}-\frac{1}{180}(\nabla^\alpha F_{\alpha\mu})^2\Bigg]. \quad (5.2)
\end{aligned}$$

However, as expected the differences amount to total derivatives only (see (A.8), (A.9)),

$$\begin{aligned}
\mathcal{L}^{(SI)}_{\text{scal}}-\mathcal{L}^{(HK)}_{\text{scal}} &= \frac{1}{16\pi^2}\frac{e^2}{m^2}\Bigg\{\frac{7}{360}\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha}) \\
&\quad +\frac{1}{180}\Big[\nabla_\alpha(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta})-\nabla_\beta(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta})\Big]\Bigg\}, \\
\mathcal{L}^{(SI)}_{\text{spin}}-\mathcal{L}^{(HK)}_{\text{spin}} &= -\frac{1}{8\pi^2}\frac{e^2}{m^2}\Bigg\{-\frac{1}{45}\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha}) \\
&\quad +\frac{1}{180}\Big[\nabla_\alpha(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta})-\nabla_\beta(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta})\Big]\Bigg\}. \quad (5.3)
\end{aligned}$$

Similarly, agreement with the Drummond-Hathrell form of the spinor loop effective action, eq. (1.6), can be seen using a different linear combination of the same total derivatives,

$$\begin{aligned}
\mathcal{L}^{(SI)}_{\text{spin}}-\mathcal{L}^{(DH)}_{\text{spin}} &= -\frac{1}{8\pi^2}\frac{e^2}{m^2}\Bigg\{\frac{1}{36}\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha}) \\
&\quad +\frac{1}{15}\Big[\nabla_\alpha(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta})-\nabla_\beta(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta})\Big]\Bigg\}. \quad (5.4)
\end{aligned}$$

For completeness, let us also give here the form of the scalar loop effective action in the Drummond-Hathrell basis:

$$\begin{aligned}
\mathcal{L}^{(DH)}_{\text{scal}} &= \mathcal{L}^{(HK)}_{\text{scal}}+\frac{1}{16\pi^2}\frac{e^2}{m^2}\Bigg\{\frac{1}{30}\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha}) \\
&\quad +\frac{1}{45}\Big[\nabla_\alpha(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta})-\nabla_\beta(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta})\Big]\Bigg\} \\
&= \frac{1}{16\pi^2}\frac{e^2}{m^2}\Bigg[\frac{1}{12}\left(\bar{\xi}+\frac{1}{12}\right)RF^2_{\mu\nu}-\frac{1}{90}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} \\
&\quad -\frac{1}{180}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}+\frac{1}{60}(\nabla^\alpha F_{\alpha\mu})^2\Bigg]. \quad (5.5)
\end{aligned}$$

The expansion (5.1) can be easily pursued to higher orders in F using the formulas of appendix B [64]. The reduction to a minimal basis of terms becomes increasingly laborious, of course.

We have also checked by an independent calculation of the a_3 coefficients in the DBC scheme that this scheme indeed reproduces the heat kernel results, $\mathcal{L}^{(DBC)}_{\text{scal,spin}} = \mathcal{L}^{(HK)}_{\text{scal,spin}}$.

As was mentioned already in the introduction, Avramidi [30, 31] has obtained Euler-Heisenberg type formulas for the heat kernel diagonal of the Laplacian on twisted spin-vector bundles for the covariantly constant case, $\nabla_\alpha F_{\mu\nu} = \nabla_\alpha R_{\mu\nu\kappa\lambda} = 0$. If we specialize our results (3.8), (4.6) to this case (this just amounts to deleting all derivative terms, and in particular removes the second integration in (3.8), (4.6)) then they should match with the result of [31] after expanding to linear order in R there. However, that result is in a rather implicit form which still requires one to perform integrals over the holonomy group for extracting individual terms in the effective action; therefore a direct comparison would be difficult and we have not attempted it here.[5]

Finally, in the flat space limit our effective Lagrangians (3.8), (4.6) yield representations of the two-derivative corrections to the scalar and spinor QED EHL's. Those representations are similar to but more compact than the ones given in [45] which correspond to a calculation in the DBC scheme.

6 Discussion

Let us summarize the information contained in our main result, the effective Lagrangians (3.8), (4.6):

1. They contain the full information on the one-loop amplitude involving N photons and one graviton, with a massive scalar or spinor in the loop, in the limit where all photon and graviton energies are small compared to the loop mass. In future work, we hope to obtain these amplitudes in an explicit form, generalizing the one found for the pure N-photon amplitudes in [7].

2. They can be used to extend the study of the modified dispersion relations for low-energy photons in an Einstein-Maxwell background, previously restricted to the weak field expansion in the electromagnetic field [21–23], to the case of strong electromagnetic fields (although one must keep in mind that for super-strong fields the one-loop approximation is expected to break down already in the pure QED case [5].).

3. It would also be straightforward to derive from (3.8), (4.6) the corresponding corrections to the imaginary part of the effective Lagrangians, in a form which modifies the Schwinger representations (1.5) by terms of order R/m^2 in the prefactors of the universal exponentials. This could be used then to calculate the pair production rates in strong electromagnetic and weak gravitational fields. We find it hard, though, to think of a realistic scenario where the R/m^2 corrections would not be negligible with respect to the leading QED term. Here it must also be mentioned that Das and Dunne [66] have shown that the simple relation between the imaginary part of the effective Lagrangian and the pair creation rate does in general not extend to the curved space case. However, this is due to effects nonperturbative in the curvature, and is not expected to happen at finite orders in an expansion in the curvature.

[5] Very recently Avramidi and Fucci [65] have used the methods of [30, 31] to obtain a more explicit representation of the heat kernel for this covariantly constant case.

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It should be emphasized that, although we have restricted ourselves here to the approximation linear in the curvature, the formalism developed in this paper applies as well to the computation of the effective action at higher orders in the curvature. The only caveat is that, starting at the quadratic level in the curvature, all of the subtleties described in section 2 will come into play, including the need for a regularization and the introduction of appropriate worldline counterterms.

Another generalization of interest would be to consider other types of particles in the loop. Presently no natural worldline representation is known for the case of a loop graviton coupled to background gravity (although such a representation can perhaps be obtained along the lines of [67]). However, worldline path integrals representing vector and antisymmetric tensor particles coupled to background gravity have been recently constructed in [41].

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A Conventions and useful formulas

The Einstein-Maxwell theory is described by

$$\Gamma[g,A] = \int d^Dx\ \sqrt{g}\left(\frac{1}{\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{A.1}$$

where the metric $g_{\mu\nu}$ has signature $(-,+,+,\dots,+)$, $g = |\det g_{\mu\nu}|$, and $\kappa^2 = 16\pi G_N$.

We use the following conventions for the curvature tensors,

$$\begin{aligned} &[\nabla_\mu,\nabla_\nu]V^\lambda = R_{\mu\nu}{}^\lambda{}_\rho V^\rho\,, \quad R_{\mu\nu} = R_{\lambda\mu}{}^\lambda{}_\nu\,, \quad R = R^\mu{}_\mu > 0 \text{ on spheres}\,,\\ &[\nabla_\mu,\nabla_\nu]\phi = iF_{\mu\nu}\phi\,, \end{aligned} \tag{A.2}$$

where V^μ is an uncharged vector and ϕ a charged scalar.

The following identities are used in the text for simplifying the various effective Lagrangians:

$$F_{\mu\alpha;\beta}\,F^{\mu\beta;\alpha} = \frac{1}{2}F_{\mu\beta;\alpha}\,F^{\mu\beta;\alpha}\,, \tag{A.3}$$

$$F_\mu{}^\alpha\,F^{\mu\beta}{}_{;\alpha\beta} = \frac{1}{2}F_{\mu\nu}\Box F^{\mu\nu}\,, \tag{A.4}$$

$$F_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\alpha\nu\beta} = \frac{1}{2}F_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\nu\alpha\beta}\,. \tag{A.5}$$

The identities (A.3)–(A.5) are simple consequences of the Bianchi identities

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0\,, \tag{A.6}$$

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0\,. \tag{A.7}$$

The following identities are needed for the comparison of the various effective Lagrangians at level RFF in section 5:

$$\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha}) = F_{\mu\nu}\Box F^{\mu\nu} + (\nabla_\alpha F_{\mu\nu})^2\,, \tag{A.8}$$

$$\begin{aligned}\nabla_\alpha\left(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta}\right) - \nabla_\beta\left(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta}\right) &= (\nabla^\alpha F_{\alpha\mu})^2 - \frac{1}{2}(\nabla_\alpha F_{\mu\nu})^2\\ &\quad+\frac{1}{2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} - R_{\mu\nu}F^{\mu\alpha}F^\nu{}_\alpha\,.\end{aligned} \tag{A.9}$$

B Properties of the field-dependent worldline Green's functions

In this appendix we collect some basic properties of the worldline Green's functions in a constant field $\mathcal{G}_B, \mathcal{G}_F$, introduced in (2.37) (see appendix B of [36] for a more thorough discussion).

$\mathcal{G}_B$ ($\mathcal{G}_F$) inverts the kinetic operator of the bosonic (fermionic) parts of the worldline action in a background field with field strength tensor $F_{\mu\nu}$. In the present conventions, the quadratic part of the action reads

$$\begin{aligned}S_0[x^\mu;F] = \frac{1}{T}\int_0^1 d\tau\,\Big[&\frac{1}{4}\dot{y}^\mu(\tau)g_{\mu\nu}(x_0)\dot{y}^\nu(\tau) + \frac{1}{2}iTy^\mu(\tau)F_{\mu\nu}(x_0)\dot{y}^\nu(\tau)\\ &+\frac{1}{2}\psi^\mu(\tau)g_{\mu\nu}(x_0)\dot{\psi}^\nu(\tau) - iT\psi^\mu(\tau)F_{\mu\nu}(x_0)\psi^\nu(\tau)\Big]\,.\end{aligned} \tag{B.1}$$

Thus formally the worldline propagators are

$$\begin{aligned}\mathcal{G}_B(\tau_1,\tau_2) &= 2\langle\tau_1\mid\left(\partial_P{}^2 - 2iFT\partial_P\right)^{-1}\mid\tau_2\rangle\,,\\ \mathcal{G}_F(\tau_1,\tau_2) &= 2\langle\tau_1\mid\left(\partial_A - 2iFT\right)^{-1}\mid\tau_2\rangle\,,\end{aligned} \tag{B.2}$$

where the subscripts "P" and "A" keep track of the boundary conditions. Explicit formulas for these Green's functions in the SI scheme were given already in (2.37),

$$\begin{aligned}\mathcal{G}_B(\tau_1,\tau_2) &= \frac{1}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1\right),\\ \mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})}\,.\end{aligned}$$

The right hand sides of these formulas are now to be understood as power series in the matrix $\mathcal{Z}_{\mu\nu} := TF_{\mu\nu}(x_0)$, where the indices are raised and lowered with $g_{\mu\nu}(x_0)$. The point of expressing $\mathcal{G}_{B,F}$ in terms of the ordinary worldline Green's functions $\dot{G}_B, G_F$ is that it allows one to avoid making a case distinction for the ordering of $\tau_{1,2}$. For our present purposes also the following derivatives of $\mathcal{G}_{B,F}$ are needed,

$$\begin{aligned}\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} - 1\right),\\ \ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= 2\delta_{12} - 2\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}\,,\\ \dot{\mathcal{G}}_F(\tau_1,\tau_2) &= 2\delta_{12} + 2iG_{F12}\frac{\mathcal{Z}}{\cos(\mathcal{Z})}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}\,.\end{aligned} \tag{B.3}$$

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It will also be convenient to list the coincidence limits of the above five functions:

$$\begin{aligned}
\mathcal{G}_B(\tau,\tau) &= \frac{1}{2\mathcal{Z}^2}\Big(\mathcal{Z}\cot(\mathcal{Z})-1\Big)\,,\\
\dot{\mathcal{G}}_B(\tau,\tau) &= i\cot(\mathcal{Z})-\frac{i}{\mathcal{Z}}\,,\\
\ddot{\mathcal{G}}_B(\tau,\tau) &= 2\delta(0)-2\mathcal{Z}\cot(\mathcal{Z})\,,\\
\mathcal{G}_F(\tau,\tau) &= -i\tan(\mathcal{Z})\,,\\
\dot{\mathcal{G}}_F(\tau,\tau) &= 2\delta(0)+2\mathcal{Z}\tan(\mathcal{Z})\,.
\end{aligned} \tag{B.4}$$

We note that $\mathcal{G}_B$ acts, like G_B, in the space of periodic functions obeying the SI condition (2.11). Its Fourier expansion therefore involves only modes orthogonal to the constant functions. This has the consequence that

$$\int_0^1 d\tau_{1,2}\,\mathcal{G}_B^{(n)}(\tau_1,\tau_2) = 0\,, \tag{B.5}$$

where n denotes any derivative of $\mathcal{G}_B$. Finally, to recover perturbative results one needs the coefficients in the expansions of $\mathcal{G}_{B,F}$ as powers of F. These expansions can be written as follows,

$$\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= -2\sum_{n=0}^{\infty}(2i\mathcal{Z})^n g_{n+2}(\tau_1-\tau_2)\,,\\
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= -2\sum_{n=0}^{\infty}(2i\mathcal{Z})^n g_{n+1}(\tau_1-\tau_2)\,,\\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= 2\delta_{12}-2\sum_{n=0}^{\infty}(2i\mathcal{Z})^n g_{n}(\tau_1-\tau_2)\,,\\
\mathcal{G}_F(\tau_1,\tau_2) &= 2\sum_{n=0}^{\infty}(2i\mathcal{Z})^n f_{n+1}(\tau_1-\tau_2)\,,\\
\dot{\mathcal{G}}_F(\tau_1,\tau_2) &= 2\delta_{12}+2\sum_{n=1}^{\infty}(2i\mathcal{Z})^n f_{n}(\tau_1-\tau_2)\,.
\end{aligned} \tag{B.6}$$

Here the coefficient functions g_n, f_n are polynomials in $\tau_1-\tau_2$ (apart from factors of $\mathrm{sign}(\tau_1-\tau_2)$). In writing these polynomials one has a choice of variables. In terms of $\tau=\tau_1-\tau_2$ one gets, by a straightforward expansion of (2.37) (see [36, 68]),

$$\begin{aligned}
g_n(\tau) &= \frac{1}{n!}\mathcal{B}_n(|\tau|)\mathrm{sign}^n(\tau)\,,\\
f_n(\tau) &= \frac{1}{2(n-1)!}\mathcal{E}_{n-1}(|\tau|)\mathrm{sign}^n(\tau)\,.
\end{aligned} \tag{B.7}$$

Here $\mathcal{B}_n$ denotes the nth Bernoulli polynomial, $\mathcal{E}_n$ the nth Euler polynomial.

Alternatively, one can also write the same coefficient functions in terms of the vacuum Green's functions [69]. Denoting by $\bar{G}$ the coordinate worldline Green's function with its coincidence limit subtracted,

$$\bar{G}(\tau) := |\tau|-\tau^2 \tag{B.8}$$

one finds

$$
\begin{aligned}
g_0(\tau) &= 1\,,\\
g_1(\tau) &= -\frac{1}{2}\dot{G}_B(\tau,0) = -\frac{1}{2}\dot{\bar{G}}(\tau)\,,\\
g_2(\tau) &= -\frac{1}{2}G_B(\tau,0) = -\frac{1}{2}\bar{G}(\tau) + \frac{1}{12}\,,\\
g_n(\tau) &= \begin{cases} \frac{B_n}{n!} + \frac{1}{2(n-1)!}\sum_{k=1}^{n/2-1} f\big(\frac{n}{2}-1,k\big)(-\bar{G})^{k+1}(\tau) & (n>2 \quad \text{even})\\ -\frac{1}{2n!}\sum_{k=1}^{(n-1)/2} f\big(\frac{n-1}{2},k\big)(k+1)\dot{\bar{G}}(\tau)(-\bar{G})^{k}(\tau) & (n>2 \quad \text{odd}) \end{cases}
\end{aligned}
\tag{B.9}
$$

and

$$
\begin{aligned}
f_1(\tau) &= \frac{1}{2}G_F(\tau,0) = \frac{1}{2}\mathrm{sign}(\tau)\,,\\
f_2(\tau) &= -\frac{1}{4}G_F(\tau,0)\dot{G}_B(\tau,0) = -\frac{1}{4}\mathrm{sign}(\tau)\dot{\bar{G}}(\tau)\,,\\
f_n(\tau) &= \begin{cases} -\frac{1}{2n!}\sum_{k=1}^{n/2} s\big(\frac{n}{2},k\big)k\,\mathrm{sign}(\tau)\dot{\bar{G}}(\tau)(-\bar{G})^{k-1}(\tau) & (n>2 \quad \text{even})\\ \frac{1}{2(n-1)!}\sum_{k=1}^{(n-1)/2} s\big(\frac{n-1}{2},k\big)\mathrm{sign}(\tau)(-\bar{G})^{k}(\tau) & (n>2 \quad \text{odd})\,. \end{cases}
\end{aligned}
\tag{B.10}
$$

Here the $f(m,k)$ are Faulhaber numbers and the $s(m,k)$ Salié numbers. Those numbers can be defined in terms of the Bernoulli numbers as [70]

$$
\begin{aligned}
f(m,k) &= (-1)^{k+1}\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{k-j}\binom{2k-2j}{k+1}\binom{2m+1}{2j+1}B_{2m-2j}\,,\\
s(m,k) &= 2(-1)^{k}\sum_{j=0}^{\lfloor (k-1)/2\rfloor}\frac{1}{2k-2j-1}\binom{2k-2j-1}{k}\binom{2m}{2j}\Big(1-2^{2m-2j}\Big)B_{2m-2j}
\end{aligned}
\tag{B.11}
$$

$(m \geq k \geq 1)$. For easy reference, let us write down the expansions (B.6) explicitly to order $O(F^2)$, using the form (B.9), (B.10) for the coefficients:

$$
\begin{aligned}
\mathcal{G}_{B12} &= \bar{G}_{B12} - \frac{1}{6} - \frac{i}{3}\dot{G}_{B12}\bar{G}_{B12}\mathcal{Z} + \left(\frac{1}{3}\bar{G}_{B12}^2 - \frac{1}{90}\right)\mathcal{Z}^2 + O(\mathcal{Z}^3)\,,\\
\dot{\mathcal{G}}_{B12} &= \dot{G}_{B12} + 2i\left(\bar{G}_{B12} - \frac{1}{6}\right)\mathcal{Z} + \frac{2}{3}\dot{G}_{B12}\bar{G}_{B12}\mathcal{Z}^2 + O(\mathcal{Z}^3)\,,\\
\ddot{\mathcal{G}}_{B12} &= 2\delta_{12} - 2 + 2i\dot{G}_{B12}\mathcal{Z} - 4\left(\bar{G}_{B12} - \frac{1}{6}\right)\mathcal{Z}^2 + O(\mathcal{Z}^3)\,,\\
\mathcal{G}_{F12} &= G_{F12} - iG_{F12}\dot{G}_{B12}\mathcal{Z} + 2G_{F12}\bar{G}_{B12}\mathcal{Z}^2 + O(\mathcal{Z}^3)\,,\\
\dot{\mathcal{G}}_{F12} &= 2\delta_{12} + 2i\mathcal{Z}G_{F12} + 2G_{F12}\dot{G}_{B12}\mathcal{Z}^2 + O(\mathcal{Z}^3)\,.
\end{aligned}
\tag{B.12}
$$

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